RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIRST SEMESTER EXAMINATION, DECEMBER 2012

FIRST YEAR

Date : 14/12/2012 **Mathematics (Honours)**

Time : 11.00 am – 3.00 pm Paper : I Full Marks : 100

Use separate answer-book for each group

Group – A

Unit-I

Question no.1 is compulsory. Answer <u>any two</u> from Q. no. 2 to Q. no. 4 and <u>any two</u> from Q. no. 5 to Q. no. 7

1. State whether the following statements are true or false. Justify your answer. 1x5
   (i) If $f : A \to B$ and $g : B \to C$ are two mappings such that $g \circ f : A \to C$ is bijective, then both *f* and *g* are surjective.
   (ii) If A, B are nonempty sets such that A x B is infinite, then both A and B are infinite.
   (iii) Number of equivalence relations on {1, 2, 3} is 4.
   (iv) If $(G, \circ)$ be a finite group with identity *e*, then there exists an integer *m* such that $a^m = e$ for all $a \in G$.
   (v) The group $(\mathbb{R}, +)$ is cyclic.

2. Find all subgroups of $(\mathbb{Z}, +)$. 5

3. (a) Let $(S, \circ)$ be a semi-group with the identity element *e* and for any two elements $a, b \in S$, each of the equations $a \circ x = b$ and $y \circ a = b$ has a solution in S. Prove that $(S, \circ)$ is a group. 3
   (b) Find the *f*-image of the element 1 and 4 if $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ & 3 & 4 & & 2 & 5 \end{pmatrix}$ is an (i) odd permutation, (ii) even permutation. 2

4. (a) Let $(G, \circ)$ be a group in which $(a \circ b)^n = a^n \circ b^n$ hold for three consecutive positive integers for all $a, b \in G$. Show that *G* is abelian. 3
   (b) 'Union of two subgroups of a group $(G, \circ)$ is not always a subgroup of *G*'-justify. 2

5. a) Let *a* be an element of a group $(G, \circ)$ and $o(a) = n$, then prove that $o(a^p) = n$ if *p* is prime to *n*. 2
   b) Let $(G, \circ)$ be a group and *H* be the subset defined by $H = \{x \in G : x \circ g = g \circ x \text{ for all } g \in G\}$. Prove that $(H, \circ)$ is a subgroup of *G*. 3

6. Let $(G, \circ)$ be an arbitrary group and *e*(*G*) be the collection of all cyclic subgroups of *G*. Consider the map $f : G \to e(G)$ defined by $f(g) = \langle g \rangle$. Is *f* necessarily (a) injective, (b) surjective? What can you say about *G* if it is known that *f* is bijective? 1+1+3

7. a) Let H be a sub-group of a group $(G, \circ)$, and $a, b \in G$. Prove that *a*H=*b*H if and only if $a^{-1}b \in H$. 3
   b) Prove that the order of each element in a finite group $(G, \circ)$ is a divisor of o(*G*). 2

Unit-II

Answer **any five** questions: 5x5=25

8. a) Prove that every infinite bounded subset of $\mathbb{R}$ has a limit point. 3
   b) Let A be an uncountable subset of $\mathbb{R}$. Prove that A has a limit point. 2

9. a) Prove that there is no least positive real number. 2

b) If x and y are real numbers with $x<y$, then prove that there exists a rational number r such that $x<r<y$. 3

10. Define an enumerable set. Prove that the set $\{x \in \mathbb{R} : 0 < x < 1\}$ is not enumerable. 1+4

11. Prove that union of a finite collection of open sets of the set of real numbers is an open set. Hence prove that a finite set is a closed set. 3+2

12. a) If S be an infinite and bounded set of reals, then show that S', the derived set of S, is bounded. 3

b) Let A, B be two sets of real numbers. Prove that $(A \cap B)' \subset A' \cap B'$. 2

13. a) Prove that every sequence of real numbers has a monotone subsequence. 3

b) Let $\{u_n\}$ be a sequence of positive real numbers such that $\lim \sqrt[n]{u_n} = l$. Prove that $\lim u_n = 0 \; if \; 0 \le l < 1$. 2

14. a) Prove that the sequence $\{u_n\}$, where $u_1=0$, $u_2=1$ and $u_{n+2}=\frac{1}{2}(u_{n+1}+u_n)$ for all $n \ge 1$ is a Cauchy sequence.

or

Prove that a Cauchy sequence of real numbers has a convergent subsequence. 2

b) Let $\{x_n\}$ be a bounded sequence of real numbers and E be the set of all subsequential limits of $\{x_n\}$. Prove that E is a non empty, closed and bounded subset of $\mathbb{R}$. 3

15. a) Let *f* be a real valued function defined on a domain $D \subset \mathbb{R}, c \in D'$ and $\lim_{x \to c} f(x)$ exists. Prove that *f* is bounded on some neighbourhood of c, where D' is the derived set of D. 2

b) Let I=(a,b) be a bounded open interval and $f: I \to \mathbb{R}$ be a monotone increasing function on I. Prove that $\lim_{x \to b} f(x) = \sup_{x \in (a,b)} f(x)$ if f is bounded above on I. 3

Group – B

Answer <u>any five</u> questions from question no. 16 to question no. 23 and <u>any five</u> from question no. 24 to question no. 31

16. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, prove that the area of the triangle formed by their bisectors and the axis of x is $\frac{\sqrt{(a-b)^2+4h^2}}{2h} \cdot \frac{ca-g^2}{ab-h^2}$. 5

17. If the tangents at P, Q of a parabola meet at a point T and S be the focus, prove that $ST^2 = SP.SQ$. 5

18. Find the locus of the point of intersection of a pair of perpendicular tangents to the conic $\frac{l}{r} = 1 + e\cos\theta$. 5

19. Reduce the equation $3(x^2+y^2)+2xy = 4\sqrt{2}(x+y)$ to its canonical form. State the nature of the conic represented by it and determine the equations of its axes. 5

20. Prove that the locus of the poles of the normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the curve $a^6y^2 - b^6x^2 = (a^2+b^2)^2 x^2y^2$. 5

21. Show that the four points $\vec{\alpha} = 6\vec{a} - 4\vec{b} + 10\vec{c}, \vec{\beta} = -5\vec{a} + 3\vec{b} - 10\vec{c}, \vec{\gamma} = 4\vec{a} - 6\vec{b} - 10\vec{c}, \vec{\delta} = 2\vec{b} + 10\vec{c}$ are coplanar.

22. Show that the solution of the equation $t\vec{r} + \vec{r} \times \vec{a} = \vec{b}$ where t is a non zero known scalar and $\vec{a}, \vec{b}$ are two known vectors can be put as $\vec{r} = \frac{1}{t^2+|\vec{a}|^2}\left(\frac{\vec{a}\bullet\vec{b}}{t}\vec{a} + t\vec{b} + \vec{a} \times \vec{b}\right)$. 5

23. Find the shortest distance in terms of L between the lines $\vec{r} = \vec{\alpha} + t\vec{\beta}$ and $\vec{r} = \vec{\gamma} + s\vec{\delta}$,
where $\vec{\alpha} = \hat{i} + 2j + 3k,\ \vec{\beta} = 2\hat{i} + 3j + 4k,$
$\vec{\gamma} = L\hat{i} + 3j + 4k$ and $\vec{\delta} = 3\hat{i} + 4j + 5k$.
For what value of L the lines are coplanar? 4+1

24. Prove that $(x+y+1)^{-4}$ is an integrating factor of $(2xy - y^2 - y)dx + (2xy - x^2 - x)dy = 0$ and hence solve it. 2+3

25. Reduce the differential equation $y = 2px - p^2 y$ to Clairaut's form by the substitution $y^2 = Y,\ x = X$ and then obtain its complete primitive and singular solution, if any. 5

26. Solve by the method of undetermined coefficients $(D^2 - 3D)y = x + e^x \sin x, D \equiv \frac{d}{dx}$. 5

27. Solve by the method of variation of parameters $\frac{d^2y}{dx^2} + K^2 y = \tan Kx$. 5

28. Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$. 5

29. Find the orthogonal trajectories of the family of co-axial circles $x^2 + y^2 + 2gx + c = 0$ where g is the parameter and C is a constant. 5

30. Show that, if y_1 and y_2 be solutions of the equation $\frac{dy}{dx} + Py = Q$ where P and Q are functions of x alone, and $y_2 = y_1 z$ then $z = 1 + ae^{\int \frac{Q}{y_1} dx}$, where a is an arbitrary constant. 5

31. Show that $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0$ is exact and solve it completely. 5